

Conformal Deformation from Normal to Hermitian Random Matrix Ensembles

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Abstract

We investigate the eigenvalues statistics of ensembles of normal random matrices when their order N tends to infinite. In the model the eigenvalues have uniform density within a region determined by a simple analytic polynomial curve. We study the conformal deformations of normal random ensembles to Hermitian random ensembles and give sufficient conditions for the latter to be a Wigner ensemble.

1 Introduction and Statement of Results

Since early fifties Hermitian random matrix theory plays an important role in the statistical description of the spectra of complex systems [1, 2]. Recently non-Hermitian random matrices have been used to treat problems in superconductor physics with columnar defects [3, 4], in quantum chaotic systems [5], and in quantum chromodynamics [6, 7].

Normal random matrix ensembles have been playing a major role in several areas such as in the study of fractional quantum Hall effect [8], quantum Hele-Shaw flows [9], integrable hierarchies [10], and integrable structure of the Dirichlet boundary problem [11, 12].

In the present work the ability of normal ensembles to be conformally deformed into Hermitian ensembles is exploited to project density of eigenvalues of non-Hermitian matrices into the real axis. The problem is addressed by using the so-called *invariant ensemble model*, characterized by the probability of finding a $N \times N$ matrix M of a class within the ensemble given by

$$P(M)dM \propto \exp \{-N\text{Tr}[V(M)]\} dM, \quad (1)$$

with the trace $\text{Tr}[V(M)]$ and the Riemann volume dM invariant under unitary transformations. The corresponding eigenvalue density, in the limit $N \rightarrow \infty$, depends on the particular form of $V(M)$. For

the Wigner ensemble of Hermitian matrices with $V(M) = \frac{1}{\sigma^2} M^* M$ (M^* is the Hermitian conjugate of M), the entrances of M are independent and identically distributed Gaussian random variables with zero mean and variance σ^2/N . The density of eigenvalues follows the Wigner semicircle law supported on $[-2\sigma, 2\sigma]$ [2]:

$$d\mu_W(x) = \frac{1}{2\pi\sigma} \sqrt{4 - x^2/\sigma^2} \chi_{[-2\sigma, 2\sigma]}(x) dx, \quad (2)$$

where $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise.

A particularly interesting potential has been put forward by Wiegmann, Zabrodin and coworkers [10, 11, 12, 13] who established a connection between normal random matrices and conformal mappings. They considered

$$V(M) = \frac{1}{t_0} (M^* M - p(M) - p(M)^*), \quad (3)$$

where

$$p(z) = \sum_{j \geq 1} t_j z^j \quad (4)$$

with $t_0 > 0$ and $t_j \in \mathbb{C}$. As $N \rightarrow \infty$, they showed, at the level of formal manipulations, that : (A) the density of eigenvalues is uniform within a simply connected domain $D \subset \mathbb{C}$ whose boundary is given by a simple analytic curve γ ; (B) the domain D is characterized by the fact that its exterior harmonic moments

$$t_j = \frac{1}{2\pi i j} \oint_{\gamma} \bar{z} z^{-j} dz, \quad j \geq 1, \quad (5)$$

where πt_0 stands for the area of D , are the coefficients of (4); and (C) the Riemann mapping from the exterior of the unit disk onto the exterior of the domain D obeys, as a function of the t_j , the equations of the integrable dispersionless Toda hierarchy.

Potentials of the form (3) give rise two sorts of mathematical problems. Except in the case of polynomial $p(z)$ of degree 2, where the domain D is bounded by an ellipse, $V(z)$ is not bounded from below and integrals with respect to (1) diverge. The other problem concerns with the fact that D may not be uniquely determined by the moments (5). From the point of view of equilibrium measures (see Section 2), a relevant fraction of eigenvalues of a M may escape to infinity or to another Riemann surface.

Recently, the results (A) and (B) have been set in a rigorous frame by Elbau and Felder [14]. To avoid the above mentioned problems, they consider the following restrictions:

Elbau-Felder Potential. If

$$p(z) = t_1 z + t_2 z^2 + \dots + t_{n+1} z^{n+1} \quad (6)$$

is an analytic polynomial of degree $n+1$ with $t_0 > 0$ and $\mathbf{t} = (t_1, \dots, t_{n+1}) \in \mathbb{C}^{n+1}$ such that $t_1 = 0$, $|t_2| < 1/2$, Elbau–Felder potential is a real-valued function on \mathbb{C} given by

$$V(z) = \frac{1}{t_0} (|z|^2 - p(z) - \overline{p(z)}) .$$

It can be shown by direct computations that $V(z)$, under the above conditions, is positive in a neighborhood of $z = 0$ and has a non-degenerate absolute minimum at $z = 0$. From now on, $V(z)$ shall stand for the Elbau-Felder potential. The problem of divergence of the integrals is solved by Elbau–Felder in a naïve way – imposing that the eigenvalues of matrices within the normal ensemble remains bounded:

Elbau-Felder Ensemble. Let $\Sigma \subset \mathbb{C}$ be the closure of a bounded open set that contains the origin and consider the following class of matrices

$$\mathcal{N}_N(\Sigma) = \{A \in \text{Mat}_{\mathbb{C}}(N) : [A, A^*] = 0, \sigma(A) \subset \Sigma\} \quad (7)$$

where $\sigma(A)$ denotes the spectrum of A . An ensemble is said to be of Elbau-Felder type of degree $n + 1$ if it fulfills conditions stated between (6) and (7). A closed polynomial curve γ of degree n can be parametrized by

$$w \mapsto h(w) = rw + \sum_{j=0}^n a_j w^{-j}, \quad |w| = 1 \quad (8)$$

for some $r > 0$ and the $a_j \in \mathbb{C}$. Elbau and Felder have shown that, as long as $|t_2| < 1/2$ and t_0 is small enough, the problem of determining the exterior moments t_j out of the curve has a unique solution for simple closed analytic polynomial curves. They give a set of equations that defines an invertible map $F : (r^2, a_0, \dots, a_n/r^n) \longrightarrow (t_0, \dots, t_{n+1})$ from $\mathbb{R} \times \mathbb{C}^{n+1}$ into itself about $(0, 0, 2\bar{t}_2, \dots, (n+1)\bar{t}_{n+1})$ (\bar{t}_j stand for the complex conjugate of the t_j). By the Euler–Lagrange variational equations, the eigenvalues density is uniform in D . We refer to Theorem 2 for a precise statement.

In the present work, we study conformal deformations of the Elbau-Felder ensembles into Hermitian ensembles. This is achieved by considering a family of polynomial curves of degree n : $w \mapsto h(w; s)$, with the $a_j(s)$ depending on a parameter $s \in (0, 1]$. This family is chosen in such a way that $h(w; 1) \equiv h(w)$ parametrizes the initial curve γ whose interior domain D supports the eigenvalues. After the construction of $h(w; s)$ the support D (resp. the harmonic moments t_j) also depends on s under $s \mapsto D(s)$ (resp. $t \mapsto t_j(s)$). To state our result, we denote by $\boldsymbol{\tau} = (\tau_1, \dots, \tau_{n+1})$ a vector on the affine space $Z \subset \mathbb{C}^{n+1}$ with $\tau_1 = 0$ and $|\tau_2| = 1$.

Theorem 1 *Consider the Elbau-Felder ensemble with $t_0 > 0$ and $s \mapsto \mathbf{t}(s) \in \mathbb{C}^{n+1}$ such that $t_1 = 0$,*

$$t_2 = \frac{\sqrt{1-s}}{2} \exp(is^{\Delta_2} \varphi),$$

and

$$t_j = s^{\Delta_j} \tau_j \quad \text{for} \quad 3 \leq j \leq n+1,$$

with $\varphi \in [0, 2\pi)$, $\tau_j \in \mathbb{C}$, $\Delta_j \geq 1$ and $s \in (0, 1]$. There exist $r_0 = r_0(\boldsymbol{\tau}) > 0$ such that for every $0 < r < r_0$:

- 1) There is a unique simple analytic closed polynomial curve $\gamma = \gamma(s, r, \boldsymbol{\tau})$ of degree n , with external harmonic moments $\mathbf{t}(s)$ and area of interior domain πt_0 with t_0 depending on $\boldsymbol{\tau}$, r and s .
- 2) The curve is parametrized by $h(w; s, r, \boldsymbol{\tau}) = rw + \sum_{j=0}^n r^j \alpha_j w^{-j}$, with $|w| = 1$ and $\alpha_j = \alpha_j(s, \boldsymbol{\tau})$ is uniquely determined by t_0 and $\mathbf{t}(s)$.
- 3) The eigenvalue density is uniform within D , the interior domain of γ , for every $s \in (0, 1]$. Moreover, if $\Delta_j > 1$ then the Elbau-Felder ensemble can be conformally deformed, as s goes to 0, into a Wigner ensemble with support on $[-2r, 2r]$.

Remark 1 As long as $0 < r < r_0$, $h(w; s, r, \boldsymbol{\tau})$ is a Riemann mapping from the exterior of the unit disk onto the exterior domain D of γ and the area πt_0 of the domain D remains positive for all $s \in (0, 1]$.

Remark 2 The assumption $|t_2| < 1/2$ in Theorem 2 breaks down when the exterior domain $D_-(s) = \mathbb{C} \setminus D(s)$ is deformed into the slit domain $\mathbb{C} \setminus [-2r, 2r]$. In Section 4 we generalize Elbau-Felder's results using Crandall–Rabinowitz bifurcation theory from simple eigenvalues (see e.g. [15]) to construct a parametrization that allow us to let $t_2 \rightarrow 1/2$ maintaining the parameter r away from 0. Elbau-Felder's parametrization, coming from the implicit function theorem applied to the map F , mentioned right below Eq. (8), defines a curve in $\mathbb{R} \times \mathbb{C}^{n+1}$ that bifurcates at $t_2 = 1/2$.

Remark 3 If $\Delta_j > 1$ then the $\alpha_j = \alpha_j(s, \boldsymbol{\tau})$ behave, for $s \rightarrow 0$, as

- (i) $\alpha_0(s) = o(s)$,
- (ii) $\alpha_1(s) = 1 - s/2$,
- (iii) $\alpha_j(s) = o(s)$ for $1 < j \leq n$.

This relations will be used to prove item 3) of Theorem 1.

Remark 4 Mashkov et al (Sec. 6 of [12]) considered a family of simple closed analytic curves $\gamma(s)$ given by an equation $P(x, y/s) = 0$, $w = x + iy$, converging to the segment of the real line $[\alpha, \beta]$ as $s \rightarrow 0$. At the level of formal manipulations, they have shown that the eigenvalue density projected into $[\alpha, \beta]$ yields

$$\rho(x) = \lim_{s \rightarrow 0} \frac{\Delta y(x; s)}{s} = \sqrt{(x - \alpha)(\beta - x)} M(x)$$

where $\Delta y(x; s)$ is the width of $D(s)$ at the x coordinate and $M(x)$ is a smooth function, regular at the edges.

This paper is organized as follows. Section 2 presents some preliminary results and introduces the two ingredients, the balayage problem and the Schwarz function, required for the proof of Theorem 1. In Section 3 we prove two auxiliary results, Propositions 1 and 2. Section 4 uses Crandall–Rabinowitz bifurcation theory from simple eigenvalues to establish a smooth inverse map F^{-1} in $\mathbb{R} \times \mathbb{C}^{n+1}$ about $t_1 = 0$ and $t_2 = 1/2$. Theorem 4 in Section 5 gives an explicit expression of the Balayage measure for the potential V . Section 6 concludes the proof of Theorem 1 based in Lemma 1 and Section 7 gives some examples. We present in Section 8 our conclusions and Lemma 2 is proved in Appendix A.

2 Basic Setting

2.1 Eigenvalue Distribution for Normal Ensembles

For normal unitarily invariant ensembles, we can write Eq. (1) in terms of the spectral coordinates. The joint probability of the eigenvalues $\{z_i\}_{i=1}^N \subset \Sigma$ of M reads

$$P_N(z_1, \dots, z_N) \propto \exp \left\{ - \left(2 \sum_{1 \leq i < j \leq N} \log |z_i - z_j|^{-1} + N \sum_{i=1}^N V(z_i) \right) \right\}. \quad (9)$$

Introducing the empirical measure of the eigenvalues

$$d\mu_N(z) = N^{-1} \sum_{i=1}^N \delta(z - z_i) d^2 z, \quad (10)$$

(9) can be written as

$$P_N(z_1, \dots, z_N) = Z_N^{-1} e^{-N^2 I^V(\mu_N)},$$

where Z_N is the normalization and

$$I^V(\mu) \equiv \int (V(z) + U^\mu(z)) d\mu(z) \quad (11)$$

is the total energy. The logarithmic potential associated with μ given by

$$U^\mu(z) \equiv \int \log |z - w|^{-1} d\mu(w) . \quad (12)$$

The integrals with respect to (9) have, in the limit $N \rightarrow \infty$, dominant contribution governed by a variational problem:

$$E^V \equiv \inf_{\mu \in \mathcal{M}(\Sigma)} I^V(\mu) , \quad (13)$$

where the infimum is taken over the set $\mathcal{M}(\Sigma)$ of Borel probability measures in $\Sigma \subset \mathbb{C}$. If a probability measure μ^V satisfying

$$E^V = I(\mu^V)$$

exists, it is called the equilibrium measure associated with V . The empirical measure (10) is known to converge weakly to a unique equilibrium measure as $N \rightarrow \infty$ (see [2] for Hermitian ensembles and [9] for normal ensembles).

Theorem 2 (Elbau-Felder) *Consider the Elbau-Felder ensemble of degree n . There is $\delta > 0$ such that for all $0 < t_0 < \delta$ a unique equilibrium measure $d\mu$ exists and is uniform within a domain $D \subset \Sigma$ that contains the origin:*

$$d\mu = \frac{1}{\pi t_0} \chi_D(z) d^2 z; \quad (14)$$

D is uniquely determined by the exterior harmonic moments ($t_1 = 0$)

$$\begin{aligned} \pi t_0 &= \int_D d^2 z \quad \text{the area of } D \\ t_k &= \frac{-1}{\pi k} \int_{\mathbb{C} \setminus D} z^{-k} d^2 z, \quad \text{if } k = 2, \dots, n+1, \\ t_k &= 0 \quad \text{if } k > n+1 \end{aligned} \quad (15)$$

and its boundary γ is a simple closed analytic polynomial curve of degree n ; if $h(w) = rw + a_0 + a_1/w + \dots + a_n/w^n$, $|w| = 1$, parametrizes γ , then

$$t_0 = r^2 - \sum_{j=1}^n j |a_j|^2. \quad (16)$$

There exist homogeneous universal polynomials $P_{jk} \in \mathbb{Z}[r, a_0, \dots, a_{k-j}]$ of degree $k - j + 1$, $1 \leq j \leq k \leq n+1$ such that

$$j t_j = \bar{a}_{j-1} r^{-j+1} + \sum_{k=j}^n \bar{a}_k r^{-k} P_{jk}(r, a_0, \dots, a_{k-j}) \quad (17)$$

is an invertible transformation from $\mathbb{R} \times \mathbb{C}^n$ into itself in a neighborhood of $(r^2, a_0, a_1/r, \dots, a_n/r^n) = (0, 0, 2\bar{t}_2, \dots, (n+1)\bar{t}_{n+1})$. For sufficiently small r , the function $h(w)$ is a Riemann mapping from the exterior of the unit disk onto the exterior of D .

The proof of existence of the equilibrium measure requires to verify Euler–Lagrange type equations. Let

$$E(z) = V(z) + \frac{2}{\pi t_0} \int_D \ln \left| \frac{z}{\zeta} - 1 \right|^{-1} d^2 \zeta \quad (18)$$

be a function defined in Σ given by V plus the logarithmic potential (12) associated with the uniform measure μ in D . Lemma 6.3 of [14] shows that $E(z) = 0$ holds for almost every $z \in D$. According to Corollary 3.5 of [14], μ is the unique equilibrium measure if

$$E(z) \geq 0 \quad \text{for every } z \in \Sigma \setminus D. \quad (19)$$

We extend Elbau–Felder’s proof to the near-slit-domains in Section 4.

2.2 Balayage Problem

We tackle the problem of analyzing conformal deformations of normal ensemble by means of balayage techniques [16]. This allows us to solve the problem focusing only on the behavior of the boundary γ of D . Therefore, in our approach the balayage technique plays a major role. Let $G \subset \overline{\mathbb{C}}$ be an open set and ∂G its boundary.

Let ν be a probability measure on G (such that $\nu(\overline{\mathbb{C}} \setminus G) = 0$) and let the logarithmic potential U^ν (see Eq. (12)) be finite and continuous on G . **The balayage problem** (or "sweeping out" problem) consists in finding a probability measure $\widehat{\nu}$ with support on ∂G such that

$$U^\nu = U^{\widehat{\nu}} \text{ almost everywhere on } \partial G. \quad (20)$$

We call $\widehat{\nu}$ the balayage measure associated with ν . Throughout this work, we consider the following space of functions:

Definition 1 *Let $G \subset \mathbb{C}$ be a bounded open set. We denote by $\mathcal{H}(G)$ the space of all holomorphic functions on G and continuous on its closure \overline{G} .*

If $G \subset \mathbb{C}$ is a bounded open set and ν is a probability measure with compact support in G , then $\widehat{\nu}$ is the unique measure supported in ∂G satisfying (20) and such that $U^{\widehat{\nu}}(z)$ is bounded in ∂G . In addition, $\widehat{\nu}$ possesses the following property (see Theorem II – 4.1 of [17]):

$$\int_G f d\nu = \int_{\partial G} f d\widehat{\nu} \quad (21)$$

holds for every $f \in \mathcal{H}(G)$.

We may choose G the interior $\dot{D} = D \setminus \gamma$ of the compact support D of the equilibrium measure μ associated with V . In this case, we write $\mu = \mu|_{\dot{D}} + \mu|_{\gamma}$ and sweep out only the part $\mu|_{\dot{D}}$ lying on G : $\widehat{\mu} = \widehat{\mu|_{\dot{D}}} + \mu|_{\gamma}$. Since the equilibrium measure has no mass concentrated in γ , we have $\mu|_{\gamma} \equiv 0$. The balayage measure associated with the equilibrium measure μ is denoted simply by $\widehat{\mu}$ and has support in γ .

2.3 Parametric Curves and Schwarz Function

The following definitions will be important for the characterization of the curves appearing in our main result. We shall start with the basic

Definition 2 *A curve Γ in \mathbb{C} is said to be simple if there exist a parametrization $t \mapsto h(t)$ for $t \in [a, b]$ such that $h(t)$ is injective, i. e., if for all $x, y \in [a, b]$ $h(x) \neq h(y)$ when $x \neq y$. If $h(a) = h(b)$, in this case Γ is said to be a simple closed curve. A curve Γ in \mathbb{C} is said to be an analytic curve if there exist a parametrization $t \mapsto h(t)$ for $t \in [a, b]$ such that h is analytic and $h'(t) \neq 0$ for $t \in [a, b]$.*

Next we introduce the polynomial curves on the complex plane.

Definition 3 A curve Γ in \mathbb{C} is said to be a polynomial curve of degree n if it is parametrically represented as

$$h(w) = rw + a_0 + a_1w^{-1} + \dots + a_nw^{-n}. \quad (22)$$

with $r > 0$, $a_n \neq 0$ and $|w| = 1$.

We shall define the Schwarz function

Definition 4 Let Γ in \mathbb{C} be an analytic arc and let Ω be a strip-like neighborhood of Γ . The Schwarz function S is the unique analytic function on Ω such that

$$S(z) = \bar{z}, \quad z \in \Gamma.$$

For a treatise on the Schwarz function with applications see [18, 16].

Remark 5 Hereafter, γ denotes a simple closed analytic polynomial curve. Moreover, S stands for the Schwarz function of γ .

Schwarz function S will play a major role in the conformal deformation of the Elbau-Felder ensemble. We shall show the balayage measure is proportional to the Schwarz function S .

3 Riemann Map

We shall prove some auxiliary results, which concern the behavior of the family $h(w; s)$ as the $\gamma(s)$ is deformed to the real line.

Proposition 1 Let $h(w; s) = rw + \sum_{j=0}^n a_j(s)w^{-j}$, $|w| = 1$, be for each $s \in (0, 1]$ the parametrization of a closed polynomial curve $\gamma(s)$ of degree n with $h(\cdot; 1) = \gamma$ a simple curve. Let the condition

$$\xi(s) := r - \sum_{j=1}^n j |a_j(s)| > 0 \quad (23)$$

be satisfied for every $s \in (0, 1]$. Then, for each $s \in (0, 1]$, the $\gamma(s)$ remains a simple polynomial curves and $h(w; s)$, as a map from the exterior of the unit disk into the exterior of $\gamma(s)$, is biholomorphic (a Riemann map). Furthermore, for every $s \in [\delta, 1]$, $0 < \delta < 1$, $t_0(s)$ is bounded away from zero.

Proof. Let us begin with the estimation of t_0 . It follows from (23) that $r > |a_j(s)|$ holds for every j . Multiplying $\xi(s)$ by r it yields that

$$0 < r^2 - \sum_{j=1}^n j |a_j(s)| r < r^2 - \sum_{j=1}^n j |a_j(s)|^2 = t_0(s)$$

is bounded away from zero for $s \in [\delta, 1]$, $0 < \delta < 1$. Eq. (23) also implies that $h(w; s)$ is an analytic curve, that is, the derivative of $h(w; s)$ with respect to w , denoted by $h'(w; s)$ is bounded away from zero:

$$|h'(w; s)| = \left| r - \sum_{j=1}^n j a_j(s) w^{-(j+1)} \right| \geq r - \sum_{j=1}^n j |a_j(s)| |w|^{-(j+1)},$$

where in the last passage we have used the triangular inequality. Since we are analyzing the exterior of the unity circle $|w| > 1$, we have

$$|h'(w; s)| \geq r - \sum_{j=1}^n j |a_j(s)| > 0 \quad \forall s \in (0, 1], \quad (24)$$

which also holds in a small neighborhood of $|w| = 1$.

Now, for every $w, z \in \mathbb{C}$ with $|w| = |z| = 1$, by the triangular inequality,

$$\begin{aligned} |h(w; s) - h(z; s)| &\geq r |w - z| - \sum_{j=1}^n |a_j(s)| \left| \frac{1}{w^j} - \frac{1}{z^j} \right| \\ &= r |w - z| - \sum_{j=1}^n |a_j(s)| |w^j - z^j| \\ &\geq \xi(s) |w - z| > 0 \end{aligned} \quad (25)$$

if $w \neq z$. The last passage follows from $|w^j - z^j| \leq j |w - z|$ which can be shown using the telescopic identity

$$w^j - z^j = w^{j-1}(w - z) + w^{j-2}(w - z)z + \cdots + (w - z)z^{j-1}$$

together with the triangular inequality. Equations (25) and (24) imply that the map $h(\cdot; s) : S^1 \rightarrow \mathbb{C}$ is an embedding, $\gamma(s)$ is a simple curve and $h(w; s)$ is a Riemann map from the exterior of the unit circle onto the exterior of $\gamma(s)$ for every $s \in (0, 1]$. The polynomial curve $\gamma(s)$ with $0 < s < 1$ preserves all properties assumed for $\gamma(1) = \gamma$, concluding the proof Proposition 1. \square

Remark 6 *Proposition 1 has an intuitive appeal. To a polynomial curve $\gamma(s)$ fail to be simple it has to develop a cusp. However, when $h(w_c, s)$ form a cusp, we have $h'(w_c, s) = 0$, this situation is prevented as long as $\xi(s) > 0$. Proposition 1 gives a sufficient condition for $h'(w, s) \neq 0$ and show that (23) is also sufficient for $\gamma(s)$ to remain a simple curve.*

The next result concerns the conditions (i – iii) of *Remark 3* and the deformation $\gamma(s)$ to the real line as $s \rightarrow 0$.

Proposition 2 *Consider a polynomial curve $\gamma(s)$ of degree n parametrized by $h(w; s)$ satisfying conditions (i – iii). Then, $\lim_{s \rightarrow 0} h(w; s)$ maps the exterior of the unit disk onto $\mathbb{C} \setminus [-2r, 2r]$.*

Proof. As $h(w, s)$ is a Riemann map for all $s \in (0, 1]$ by Proposition 1, then it suffices to show that $\lim_{s \rightarrow 0} \gamma(s) = [-2r, 2r]$. Indeed, since $|w| = 1$ we may choose $w = e^{i\theta}$ with $\theta \in [0, 2\pi]$. The Riemann map reads

$$h(e^{i\theta}; s) = re^{i\theta} + a_0(s) + a_1(s)e^{-i\theta} + \sum_{j=2}^n a_j(s)e^{-ij\theta},$$

which, under the hypotheses, yields

$$h(w; s) = r(e^{i\theta} + e^{-i\theta}) + o(s),$$

implying that $\lim_{s \rightarrow 0} h(e^{i\theta}, s) = 2r \cos \theta \in [-2r, 2r]$ for all $\theta \in [0, 2\pi]$. \square

4 Exterior Harmonic Moments of Near-to-Slit Domains

Let $\mathbf{t} = (t_1, t_2, \dots, t_{n+1})$ be the exterior harmonic moment of the domain D – containing the origin and bounded by γ – and let πt_0 be the area of D .

When a given collection \mathbf{t} of $n + 1$ complex numbers, together with $t_0 > 0$, determines a simple polynomial curve γ of degree n ? We refer to Theorem 5.3 of [14] for the solution to this moment problem. If $t_1 = 0$ and $|t_2| < 1/2$, then every complex numbers t_2, \dots, t_{n+1} determine a curve γ with these properties provided t_0 is sufficiently small. We shall give a proof of this result in a language more appropriated for the generalization needed.

The map $(\rho, \boldsymbol{\alpha}) \in \mathbb{R}_+ \times \mathbb{C}^{n+1} \mapsto (t_0, \mathbf{t}) \in \mathbb{R}_+ \times \mathbb{C}^{n+1}$ defined by (16) and by the contour integral (5),

$$\begin{aligned} jt_j &= \frac{1}{2\pi i} \oint_{|w|=1} \bar{h}(w^{-1}) h'(w) h^{-j}(w) dw \\ &= \sum_{k=j-1}^n \bar{\alpha}_k \text{Res} \left(w^{k-j} \frac{1 - \sum_{l=1}^n l \alpha_l \rho^l / w^{l+1}}{(1 + \sum_{l=0}^n \alpha_l \rho^l / w^{l+1})^j}; \infty \right) \end{aligned} \quad (26)$$

taking residues at infinity:

$$\begin{aligned} t_0 &= \rho - \sum_{j=1}^n j |\alpha_j|^2 \rho^j \\ t_j &= \frac{1}{j} \bar{\alpha}_{j-1} - \bar{\alpha}_j \alpha_0 - \left(1 + \frac{1}{j}\right) \alpha_1 \bar{\alpha}_{j+1} \rho + O(\rho^2), \quad 1 \leq j \leq n+1 \end{aligned} \quad (27)$$

with $\rho = r^2$, $\alpha_j = r^{-j}a_j$, $0 \leq j \leq n$, and $\alpha_j = 0$ if $j > n$, has a smooth inverse in a neighborhood of $(0, 0, t_2, \dots, t_{n+1})$ provided $|t_2| \neq 1/2$.

In this section, we show how the inverse function theorem is applied in this situation and extend it for the case $|t_2| = 1/2$. We also verify whether the inverse determines a simple polynomial curve $\gamma = \partial D$ and whether a measure μ , uniform in a near-to-slit domain D , is the equilibrium measure of the Elbau-Felder ensemble.

4.1 Bifurcating Curves

We observe that $\rho^* = \alpha_0^* = 0$ and $\alpha_j^* = (j+1)\bar{t}_{j+1}$, $j = 1, \dots, n$ solve the equations (27) for $(\rho, \alpha_0, \dots, \alpha_n)$ and because it takes complex conjugation of the t_{j+1} , we look (27) as a map from $\mathbb{R}_+ \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ into itself

$$F : (\rho, \boldsymbol{\alpha}, \bar{\boldsymbol{\alpha}}) \longmapsto (t_0, \mathbf{t}, \bar{\mathbf{t}}) \quad (28)$$

and we write

$$\begin{aligned} 0 &= \frac{F(\rho, \boldsymbol{\alpha}^* + \rho\boldsymbol{\varphi}, \bar{\boldsymbol{\alpha}}^* + \rho\bar{\boldsymbol{\varphi}}) - (t_0, \mathbf{t}, \bar{\mathbf{t}})}{\rho} \\ &= l(1, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}}) + p(\rho, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}}) - (\tau_0, \mathbf{0}, \mathbf{0}) . \end{aligned} \quad (29)$$

Here, F has been expanded in Taylor series about $(\rho^*, \boldsymbol{\alpha}^*, \bar{\boldsymbol{\alpha}}^*)$ with remainder $\rho p(\rho, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}}) = O(\rho^2)$, $\tau_0 = t_0/\rho = O(1)$ and l is the linear map

$$l(1, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}}) = L[\rho^*, \boldsymbol{\alpha}^*, \bar{\boldsymbol{\alpha}}^*] \begin{pmatrix} 1 \\ \boldsymbol{\varphi} \\ \bar{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} 1 - 4|t_2|^2 & \mathbf{0}^T & \mathbf{0}^T \\ -\bar{\mathbf{v}} & -\bar{K} & J^{-1} \\ -\mathbf{v} & J^{-1} & -K \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{\varphi} \\ \bar{\boldsymbol{\varphi}} \end{pmatrix} \quad (30)$$

where $\mathbf{0}$ is the null column vector in \mathbb{C}^{n+1} , $\bar{v}_j = 2(1 + 1/j)(j+2)\bar{t}_2 t_{j+2}$ if $1 \leq j < n$ with $v_n = v_{n+1} = 0$, O denotes the $(n+1) \times (n+1)$ null matrix, $J = \text{diag}\{j\}_{j=1}^{n+1}$ and K is a $(n+1) \times (n+1)$ matrix with $\bar{K}_{i1} = (i+1)t_{i+1}$ for $1 \leq i \leq n$ and 0 otherwise.

Since L is invertible for $|t_2| \neq 1/2$, (28) has a smooth inverse defined in a neighborhood of $(t_0, \mathbf{t}) = (0, 0, t_2, \dots, t_{n+1})$. The implicit function theorem applied to (29) (with $\mathbf{t} \in \mathbb{C}^{n+1}$ fixed) uniquely defines two smooth curves parametrized by ρ :

$$\boldsymbol{\varphi}(\rho) = T\mathbf{v} + B\bar{\mathbf{v}} + O(\rho)$$

$$\tau_0(\rho) = 1 - 4|t_2|^2 + O(\rho)$$

on \mathbb{C}^{n+1} and \mathbb{R}_+ , respectively, where $B = (1 - 4|t_2|^2)^{-1}JK$ and $T = J + 2\bar{t}_2 B$. We note that the leading order in ρ of equation (29) can be written as

$$\begin{aligned} 1 - 4|t_2|^2 &= \tau_0 \\ \begin{pmatrix} -\bar{K} & J^{-1} \\ J^{-1} & -K \end{pmatrix} \begin{pmatrix} \boldsymbol{\varphi} \\ \bar{\boldsymbol{\varphi}} \end{pmatrix} &= \begin{pmatrix} \bar{\mathbf{v}} \\ \mathbf{v} \end{pmatrix} \end{aligned} \quad (31)$$

whose solution is given in Appendix A.

The function $t_0 = \rho\tau_0(\rho)$ is monotone increasing in $\rho \in [0, a]$ for $|t_2| < 1/2$ and sufficiently small a , and its inverse, $\rho(t_0, \mathbf{t})$, is a well defined function of t_0 and \mathbf{t} . The inverse of (27) in $\mathbb{R}_+ \times \mathbb{C}^{n+1}$ thus reads

$$(t_0, \mathbf{t}) \longmapsto (\rho(t_0, \mathbf{t}), \boldsymbol{\alpha}^* + \rho(t_0, \mathbf{t}) \boldsymbol{\varphi} \circ \rho(t_0, \mathbf{t})) . \quad (32)$$

The above application of the implicit function theorem breaks down if the second harmonic moment t_2 tends to $1/2$ and this is the case when the external domain of $\gamma(s)$ tends, as $s \rightarrow 0$, to the slit domain $\mathbb{C}/[-2r, 2r]$ (see Proposition 2). We need, therefore, to extend Theorem 5.3 of [14] to include this case. For this, we shall rescale all components of \mathbf{t} , excepted t_2 whose modulus square will be denoted by $\lambda = |t_2|^2$, as

$$t_j = (1 - 4\lambda)\tau_j, \quad j \neq 2 \quad (33)$$

and apply the constructive bifurcation theory from a simple eigenvalue developed by Crandall and Rabinowitz (see e.g. [15]). The theory applied to equation (29) uses the implicit function theorem with the role of ρ and λ exchanged. Instead of the two parametric curves $\varphi_j = \varphi_j(\rho)$ and $\tau_0 = \tau_0(\rho)$, we consider $\tilde{\varphi}_j = \tilde{\varphi}_j(\rho, \lambda)$ and $\tilde{\tau}_0 = \tilde{\tau}_0(\rho, \lambda)$ as a function of λ for $(\rho, \boldsymbol{\tau}) \in \mathbb{C}^{n+1}$ fixed, where $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_{n+1})$ is a vector in the affine space of \mathbb{C}^{n+1} with $|\tau_2| = 1$, denoted by Z .

We observe that $(\rho^*, \boldsymbol{\alpha}^*)$ with $|t_2| = 1/2$ is a bifurcation point for (27) since every neighborhood of this point contains a solution which differs from (32). Note that the tangent map of (27) at $(\rho^*, \boldsymbol{\alpha}^*)$ with $|t_2| = 1/2$ is singular i. e., L is not invertible at $\lambda = |t_2|^2 = 1/4$. Using bifurcation theory, we shall construct a pair of smooth curves for fixed $(\rho, \boldsymbol{\tau}) \in \mathbb{C}^{n+1}$: $\tilde{\varphi}_j = \tilde{\varphi}_j(\lambda)$ and $\tilde{\tau}_0 = \tilde{\tau}_0(\lambda)$, $\lambda \in (1/4 - b, 1/4]$ for some $b > 0$, such that $\tilde{\varphi}_j(1/4) = \varphi_j(\rho = 0, |t_2| = 1/2)$ and $\tilde{\tau}_0(1/4) = \tau_0(\rho = 0, |t_2| = 1/2)$.

Proposition 3 *Given $(\rho, \boldsymbol{\tau}) \in \mathbb{R}_+ \times Z \simeq \mathbb{C}^{n+1}$, there exist two uniquely defined smooth curves, $\lambda \mapsto \boldsymbol{\zeta}(\rho, \lambda, \boldsymbol{\tau})$ on \mathbb{C}^{n+1} and $\lambda \mapsto \tilde{\tau}_0(\rho, \lambda, \boldsymbol{\tau})$ on \mathbb{R} , defined by (38), such that the inverse of (27) in a neighborhood of $(0, 0, \boldsymbol{\tau})$ in $\mathbb{R}_+ \times \mathbb{R}_+ \times Z$ is written as*

$$(t_0, \rho, \boldsymbol{\tau}) \longmapsto (\rho, \boldsymbol{\alpha}^* \circ \lambda(t_0, \rho, \boldsymbol{\tau}) + \rho(\boldsymbol{\alpha}_0 + \boldsymbol{\zeta} \circ \lambda(t_0, \rho, \boldsymbol{\tau}))) \quad (34)$$

where $\lambda(t_0, \rho, \boldsymbol{\tau})$ is a function of t_0, ρ and $\boldsymbol{\tau}$ which is the unique solution for λ of $t_0 = \rho\tilde{\tau}_0(\rho, \lambda, \boldsymbol{\tau})$ in the domain $0 \leq \lambda \leq 1/4$, $\rho \in [0, \hat{r}^2]$ with $\hat{r} = \hat{r}(\boldsymbol{\tau})$ sufficiently small. Moreover, there exist $\bar{r} = \bar{r}(\boldsymbol{\tau}) > 0$ such that $h(w)$, defined by (8) with $r = \sqrt{\rho}$ and $a_j = r\alpha_j$ given by (34), parametrizes a simple polynomial curve of order n which can be deformed to the segment $[-2r, 2r]$, for every fixed $r < \bar{r}$.

Proof. It suffices, for the first statement, to verify the hypothesis of Theorem 1 in Sec. 3 of [15]. For this, it is convenient to write (29) as

$$l_0(1, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}}) + (1 - 4\lambda)l_1(1, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}}) + \tilde{p}(\lambda, \rho, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}}) - (\tau_0, \mathbf{0}, \mathbf{0}) \quad (35)$$

where l_0 and l_1 are linear maps in $\mathbb{R}_+ \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$, with $l_0 = l|_{\lambda=1/4}$ and $4l_1 = -\partial(l+p)/\partial\lambda|_{\lambda=1/4}$:

$$l_0(1, \varphi, \bar{\varphi}) = L_0 \begin{pmatrix} 1 \\ \varphi \\ \bar{\varphi} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & -\bar{K}_0 & J^{-1} \\ \mathbf{0} & J^{-1} & -K_0 \end{pmatrix} \begin{pmatrix} 1 \\ \varphi \\ \bar{\varphi} \end{pmatrix} \quad (36)$$

$$l_1(1, \varphi, \bar{\varphi}) = L_1 \begin{pmatrix} 1 \\ \varphi \\ \bar{\varphi} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^T & \mathbf{0}^T \\ -\bar{\mathbf{v}}_1 & -\bar{K}_1 & O \\ -\mathbf{v}_1 & O & -K_1 \end{pmatrix} \begin{pmatrix} 1 \\ \varphi \\ \bar{\varphi} \end{pmatrix} \quad (37)$$

with $(\bar{\mathbf{v}}_1)_j = (1 + 1/j)(j+2)\tau_{j+2}/\tau_2 + O(\rho)$, $(K_0)_{ij} = \tau_2\delta_{i1}\delta_{j1} + O(\rho)$ and, given

$$\bar{\mathbf{w}}_1 = (-\tau_2/2, 3\tau_3, \dots, (n+1)\tau_{n+1}, 0) ,$$

K_1 is, up to the leading order in ρ , a $(n+1) \times (n+1)$ matrix with \mathbf{w}_1 in its first column and 0 everywhere else. Moreover, $\tilde{p}(\lambda, \rho, \varphi, \bar{\varphi})$ is a smooth map from $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ to $\mathbb{R}_+ \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ which satisfies

$$\tilde{p}(1/4, \rho, \varphi, \bar{\varphi}) = \tilde{p}(\lambda, 0, \varphi, \bar{\varphi}) = \frac{\partial \tilde{p}}{\partial \lambda}(1/4, \rho, \varphi, \bar{\varphi}) = 0 .$$

We observe that (29), and consequently (35), extends by continuity to $\rho = 0$. In this way, we define

$$G(\rho, \lambda, \zeta, \bar{\zeta}) = \frac{F(\rho, \boldsymbol{\alpha}^* + \rho(\boldsymbol{\alpha}_0 + \zeta), \bar{\boldsymbol{\alpha}}^* + \rho(\bar{\boldsymbol{\alpha}}_0 + \bar{\zeta})) - (t_0, \mathbf{t}, \bar{\mathbf{t}})}{\rho} ,$$

for $(\lambda, \rho, \zeta, \bar{\zeta}) \in [1/4 - b, 1/4 + b] \times [0, a] \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ where, for $\boldsymbol{\alpha}_0 = (\sqrt{\tau_2}, 0, \dots, 0)$, $(1, \boldsymbol{\alpha}_0, \bar{\boldsymbol{\alpha}}_0)$ is an eigenvector of L_0 associated with the (simple) null eigenvalue. It follows by (35)-(37), that

$$G(\rho, \lambda, \zeta, \bar{\zeta}) = l_0(0, \zeta, \bar{\zeta}) + (1 - 4\lambda)l_1(1, \boldsymbol{\alpha}_0 + \zeta, \bar{\boldsymbol{\alpha}}_0 + \bar{\zeta}) + \tilde{p}(\lambda, \rho, \boldsymbol{\alpha}_0 + \zeta, \bar{\boldsymbol{\alpha}}_0 + \bar{\zeta}) - (\tau_0, \mathbf{0}, \mathbf{0}) .$$

and the implicit function theory can be applied to $G = 0$ provided the derivative of $G(\rho, \lambda, \zeta, \bar{\zeta})$ about $(\lambda, \zeta, \bar{\zeta}) = (1/4, \mathbf{0}, \mathbf{0})$, defined by the linear map

$$(\lambda, \zeta, \bar{\zeta}) \mapsto l_0(0, \zeta, \bar{\zeta}) - 4\lambda l_1(1, \boldsymbol{\alpha}_0, \bar{\boldsymbol{\alpha}}_0) = \begin{pmatrix} -4 & \mathbf{0}^T & \mathbf{0}^T \\ 4(\bar{\mathbf{v}}_1 + \bar{\mathbf{w}}_1/\sqrt{\tau_2}) & -\bar{K}_0 & J^{-1} \\ 4(\mathbf{v}_1 + \mathbf{w}_1/\sqrt{\tau_2}) & J^{-1} & -K_0 \end{pmatrix} \begin{pmatrix} \lambda \\ \zeta \\ \bar{\zeta} \end{pmatrix} ,$$

is nonsingular (see Theorem 1 in Sec. 3 of [15]). Since it is always invertible, $G = 0$ (with $\rho > 0$ and $\boldsymbol{\tau} \in Z$ fixed) uniquely defines two smooth parametric curves:

$$\begin{aligned} \zeta(\lambda) &= \bar{\tau}_2 K_0(\mathbf{v}_1 + \mathbf{w}_1/\sqrt{\tau_2}) + K_0(\bar{\mathbf{v}}_1 + \bar{\mathbf{w}}_1/\sqrt{\tau_2}) + O(1 - 4\lambda, \rho) \\ \tilde{\tau}_0(\lambda) &= 1 - 4\lambda + O((1 - 4\lambda)^2, \rho) \end{aligned} \quad (38)$$

on \mathbb{C}^{n+1} and on \mathbb{R}_+ . As in the previous case, writing $\eta = 1 - 4\lambda$, the leading order of equation $G = 0$ reads

$$\eta = \tilde{\tau}_0$$

$$\begin{pmatrix} -\bar{K}_0 - \eta \bar{K}_1 & J^{-1} \\ J^{-1} & -K_0 - \eta K_1 \end{pmatrix} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} = \eta \begin{pmatrix} \bar{\mathbf{v}}_1 + \bar{\mathbf{w}}_1 \sqrt{\tau_2} \\ \mathbf{v}_1 + \mathbf{w}_1 / \sqrt{\tau_2} \end{pmatrix} \quad (39)$$

from where (38) can be readily obtained.

The function $t_0 = \rho \tilde{\tau}_0(\rho, \lambda)$ is monotone decreasing in λ , for $0 < \lambda \leq 1/4$, $\rho \in [0, \hat{r}^2]$ with \hat{r} sufficiently small, and its inverse, $\lambda(t_0, \rho, \boldsymbol{\tau})$, is a well defined function of t_0, ρ and $\boldsymbol{\tau}$. The inverse of (27) in $\mathbb{R}_+ \times \mathbb{R}_+ \times \hat{Z}$ is thus given by (34). By (41), $\hat{r} = \hat{r}(\boldsymbol{\tau})$ is defined by

$$2 \sum_{j=1}^n j(j+1)^2 |\tau_{j+1}|^2 \hat{r}^{2(j-1)} = 1.$$

To conclude the proof, it remains to verify that

$$h(w) = rw + \alpha_0 + r \frac{\alpha_1}{w} + \dots + r^n \frac{\alpha_n}{w^n}, \quad |w| = 1 \quad (40)$$

with

$$\alpha_j = \alpha_j^* \circ \lambda(t_0, \rho, \boldsymbol{\tau}) + \rho(\alpha_{0,j} + \zeta_j \circ \lambda(t_0, \rho, \boldsymbol{\tau})) \quad (41)$$

parametrizes a simple polynomial curve $\gamma = \gamma(t_0)$. For this, in order γ to approach the segment $I = [-2r, 2r]$, as the area of its interior πt_0 tends to 0, we set $\tau_2 = 1$ and let τ_3 and $\rho = r^2$ so that

$$\alpha_0 = \rho(\alpha_{0,1} + \zeta_1) = 12\Re(\tau_3)r^2 + O(r^4, t_0) \quad (42)$$

remains inside I , closed to the origin. Now, it is enough to verify the hypothesis (23) of Proposition 1. It follows immediately from

$$\alpha_1 = \left(1 - \frac{t_0}{2r^2}\right) + O(t_0^2)$$

$$\alpha_j = \frac{t_0}{r^2}(j+1)\bar{\tau}_{j+1} + O(t_0^2), \quad 2 \leq j \leq n \quad (43)$$

for every $\boldsymbol{\tau}$, that

$$\xi = r - \sum_{j=1}^n jr^j |\alpha_j| \geq \frac{t_0}{r^2} \left(\frac{r}{2} - \sum_{j=2}^n j(j+1)r^j |\tau_{j+1}| \right) > 0$$

holds for $r < \bar{r}$ where \bar{r} is defined by equating the expression between parenthesis to 0. \square

Remark 7 r_0 in Theorem 1 is thus given by $\min(\bar{r}, \hat{r})$.

Remark 8 The results of Proposition 3 can be adapted for (33) scaled with different power of $(1 - 4\lambda)$: $t_j = (1 - 4\lambda)^{\Delta_j} \tau_j$, $\Delta_j \geq 1$ for each $j \neq 2$.

Note that Proposition 3 holds for simple closed analytic curves provided l, l_0 and l_1 are Frechét derivatives of F with respect to an appropriate Banach space.

4.2 Equilibrium Measure

Equation (19) is equivalent to

$$0 \leq \mathcal{E}(w) := E(h(w)) = \frac{1}{t_0} \left(|h(w)|^2 - |h(1)|^2 + 2\Re \int_1^w \bar{h}(\zeta^{-1}) h'(\zeta) d\zeta \right) \quad (44)$$

for $w \in h^{-1}(\Sigma)$ such that $|w| \geq 1$ (see eq. (16) of [14]), and it suffices to verify only for $|w| \geq 1/R$, where $R = \max \{|w| : h'(w) = 0, w \in \mathbb{C}\}$ is the critical radius of γ , and r sufficiently small.

Let $h^{(0)}(w) = rw + \alpha_0 + r\alpha_1/w$ be the parametrization of an ellipse that approximates γ and let $(t_0^{(0)}, t_2^{(0)})$ denote the corresponding external harmonic moment. We denote by $\mathcal{E}^{(0)}(w)$ the function defined in (44) for the ellipse and note that, by Subsection 6.2 of [14], $t_0^{(0)} \mathcal{E}^{(0)}(w)$ remains bounded for $w \in h^{-1}(\Sigma)$ such that $|w| \geq 1$. For $(1 - t_0/r^2)^{-1} \leq |w| < r^{-\alpha}$, $0 < \alpha < 1/3$, we have

$$\begin{aligned} \frac{1}{t_0} (h(w^{-1}) - h^{(0)}(w^{-1})) h'(w) &= \frac{1}{t_0} \frac{r^2 \alpha_2}{w^{-2}} r + O(t_0) = O(r^{1-2\alpha}) \\ \frac{1}{t_0} h^{(0)}(w^{-1}) (h'(w) - h^{(0)'}(w)) &= \frac{1}{t_0} r \alpha_1 w \frac{-2r^2 \alpha_2}{w^3} + O(t_0) = O(r) \\ \frac{t_0 - t_0^{(0)}}{t_0^2} &= \frac{1}{t_0^2} r^4 \alpha_2^2 + O(t_0) = O(1) \\ t_0 \mathcal{E}^{(0)}(w) &= O(r^{2-\alpha}) \end{aligned}$$

uniformly in t_0 as $t_0 \rightarrow 0$. Consequently,

$$\begin{aligned} |\mathcal{E}(w)| &\geq \frac{1}{t_0} |t_0 \mathcal{E}^{(0)}(w)| - |\mathcal{E}(w) - \mathcal{E}^{(0)}(w)| \\ &\geq \frac{1}{t_0} |t_0 \mathcal{E}^{(0)}(w)| - \frac{1}{t_0} \left| |h(w)|^2 - |h^{(0)}(w)|^2 - |h(1)|^2 + |h^{(0)}(1)|^2 \right| \\ &\quad - \frac{2}{t_0} \left| \Re \int_1^w (h(\zeta^{-1}) - h^{(0)}(\zeta^{-1})) h'(\zeta) d\zeta \right| \\ &\quad - \frac{2}{t_0} \left| \Re \int_1^w h(\zeta^{-1}) (h'(\zeta) - h^{(0)'}(\zeta)) d\zeta \right| - \left| \frac{t_0 - t_0^{(0)}}{t_0^2} t_0 \mathcal{E}^{(0)}(w) \right| \end{aligned}$$

is strictly positive for t_0 sufficiently small. For $w \in h^{-1}(\Sigma)$ such that $|w| \geq r^{-\alpha}$ we may proceed exactly as in [14].

5 Explicit Balayage Measures

To explicitly obtain the balayage measure in terms of the potentials we need an auxiliary result.

Theorem 3 *If $V : \Sigma \rightarrow \mathbb{R}$ is a potential defined in a compact set $\Sigma \subset \mathbb{C}$ with continuous second partial derivatives in its interior, then the variational problem (13) is attained at a unique equilibrium measure μ^V supported in a compact set $D \subset \Sigma$ given by*

$$d\mu^V(z) = \frac{1}{4\pi} \Delta V d^2z, \quad (45)$$

at almost every (w.r.t. the Lebesgue measure d^2z) interior point of D , where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator.

Proof. Using the smoothness of V the proof follows from Theorems I – 1.3 pp. 27 and II – 1.3 pp. 85 of [17]. \square

We derive an explicit equation for the balayage measure in terms of the potential V . The result will be of fundamental importance to establish the conformal deformation of the Elbau-Felder ensemble.

A direct application of this theorem to the potential V shows that the density of eigenvalues is indeed uniform within D . The Elbau-Felder potential reads $V(z) = (z\bar{z} - p(z) - \overline{p(z)})/t_0$, remembering that $\Delta = 4\partial_z\partial_{\bar{z}}$ and $p(z)$ is analytic ($\partial_{\bar{z}}p(z) = \partial_z\overline{p(z)} = 0$), it follows that $d\mu(z) = 1/(\pi t_0)d^2z$ within D . Moreover, we have the following result

Theorem 4 *Let V be an Elbau–Felder potential. The balayage measure $\hat{\mu}$ associated with the equilibrium measure μ^V with support on $\gamma = \partial D$ is*

$$d\hat{\mu}(z) = \frac{1}{2\pi i t_0} S(z) dz, \quad (46)$$

where dz is the measure of arc length on γ .

Proof. The main ingredient is the Green theorem. Our hypotheses on V , D and ∂D guarantee that the Green theorem is applicable. Using that $\Delta = 4\partial_z\partial_{\bar{z}}$ and $f \in \mathcal{H}(D)$, by (45) and applying the Green theorem we have

$$\begin{aligned} \int_D f(z) d\mu^V(z) &= \frac{1}{\pi} \int_D \partial_{\bar{z}}(f(z) \partial_z V(z)) d^2z \\ &= \frac{1}{2i\pi} \int_{\partial D} f(z) \partial_z V(z) dz \end{aligned}$$

and by the balayage measure property Eq. (21), we have

$$\int_{\partial D} f(z) d\hat{\mu}^V(z) = \frac{1}{2i\pi} \int_{\partial D} f(z) \partial_z V(z) dz$$

holds for every continuous function on ∂D , from where it follows the continuity of the balayage measure with respect to the Lebesgue measure. Next, we have that

$$d\hat{\mu} = \frac{1}{2\pi i} \partial_z V(z) = \frac{1}{t_0} \left(\bar{z} - \sum_{k=2}^{n+1} k t_k z^{k-1} \right)$$

the term of the sum does not contribute to a contour integral with respect to $\widehat{\mu}$ for test functions $f \in \mathcal{H}(D)$ (by continuity and by the Cauchy theorem). Then, by using the definition $\bar{z} = S(z)$, $z \in \gamma$, of the Schwarz function we obtain Eq. (46), concluding the proof. \square

Eq. (46) relates the equilibrium measure with the Schwarz function of boundary of the support. We shall conclude, by applying the Cauchy theorem, that only the branch cut in the interior domain D of the Schwarz function will contribute to the balayage measure. Thus, questions concerning the equilibrium measure μ turns to the behavior of the Schwarz function. It turns out that, except when γ is a line or a circle arc, the Schwarz function S always has a branch cut [18]. Our next result draws some conclusions on the behavior of the branch cuts.

Proposition 4 *If γ is a simple closed analytic curve, then the Schwarz function S associated with γ must have branching points in its interior.*

Proof. Take $f \in \mathcal{H}(D)$ such that $\int_D f(z)d\mu \neq 0$. By the property (21) of balayage measure together with the Theorem 4, we have

$$\int_D f(z)d\mu(z) = \frac{1}{2\pi i t_0} \oint_{\gamma} f(z)S(z)dz.$$

Suppose now that S has no branch point in the interior of γ , then by using the Cauchy theorem we would then conclude, contrarily to the hypothesis, that $\int_D f(z)d\mu = 0$. Therefore, we must have an even number of branch points inside D . Another important result is that the branch point never touches the curve γ . This can be proved by noting that γ is, by hypothesis, a simple analytic polynomial curve and the Schwarz function S must be analytic on γ and on its neighborhood (see [14, 18]), what exclude the case of the branch point touching the curve γ , concluding the proof. \square

6 Conformal Deformation

In this section we conclude the proof of our main result, Theorem 1. Since the Riemann map $h(z; s)$ is conformal from the exterior of the unit disk onto the exterior of $\gamma(s)$ it has a well defined inverse from the exterior of $\gamma(s)$ onto the exterior of the unit disk. We shall denote its inverse by $H(z; s)$:

$$h(H(z; s); s) = z \tag{47}$$

for all z in the exterior of $\gamma(s)$ and $s \in (0, 1]$.

The Schwarz functions S can be related to the Riemann map h and its inverse H by the following:

Proposition 5 *Let γ be a polynomial curve parametrized by h . Then the Schwarz function is a biholomorphic map in a neighborhood of γ and is given by*

$$S(z) = \bar{h} \left(\frac{1}{H(z)} \right),$$

where $\bar{h}(w) = rw + \bar{a}_0 + \bar{a}_1 w^{-1} + \dots + \bar{a}_n w^{-n}$.

The proof can be found in Refs. [14, 18].

Proposition 6 *If $h(z; s)$ satisfies the conditions (i – iii) of Remark 3, then the inverse function $H(z; s)$ of $h(z; s)$ reads*

$$H(z; s) = \frac{z + \sqrt{z^2 - 4ra_1(s)}}{2r} + o(s) \quad (48)$$

for $0 < s < \varepsilon$ and ε small enough.

Proof. We have by Eqs. (22) and (47)

$$h(H(z; s); s) = rH(z; s) + \sum_{j=0}^n a_j(s)H(z; s)^{-j} = z$$

which yields

$$rH^2(z; s) - (z + o(s))H(z; s) + a_1(s) = 0. \quad (49)$$

Solving the equation for H the result follows. Note that $|H| = \mathcal{O}(1)$ in the neighborhood of $\gamma(s)$. \square

Remark 9 *Equation (48) has branches, namely the minus square root, which can be directly seen from Eq. (49). We do not consider the minus square root, because it does not yield $S(z) = \bar{z}$ on $\gamma(s)$.*

Lemma 1 *Let $d\hat{\mu}(z; s)$ be the balayage measure supported on $\gamma(s)$ of the Elbau-Felder ensemble. Assume the conditions (i–iii) to hold. Then, given ε sufficiently small, for $0 < s < \varepsilon$ and $f \in \mathcal{H}(D(s))$, we have*

$$\oint_{\gamma(s)} f(z) d\hat{\mu}(z; s) = \frac{1}{2\pi r^2} \int_{-2r}^{2r} f(z) \sqrt{4r^2 - x^2} dx + o(1). \quad (50)$$

Proof. Given z in a neighborhood of $\gamma(s)$ and $\varepsilon > 0$ sufficiently small, the Schwarz function of $\gamma(s)$ for $0 < s < \varepsilon$ by Proposition 5 reads

$$S(z; s) = r \frac{1}{H(z; s)} + \bar{a}_1(s)H(z; s) + \dots + \bar{a}_n(s)H^n(z; s)$$

which together with (48), yields

$$S(z; s) = Ez + \Lambda \sqrt{z^2 - 4ra_1(s)} + g(z; s) \quad (51)$$

where $E = (r^2 + |a_1^2(s)|)/2ra_1(s)$, $\Lambda = (|a_1^2(s)| - r^2)/2ra_1(s)$, and $g(z; s)$ is $o(s)$. By (46), we have

$$I(s) = \oint_{\gamma(s)} f(z) d\hat{\mu} = \oint_{\gamma(s)} f(z) \frac{S(z; s)}{2\pi i t_0(s)} dz,$$

for $f \in \mathcal{H}(D(s))$, which together with (51), gives

$$I(s) = \oint_{\gamma(s)} \frac{f(z)}{2\pi i t_0(s)} \left(Ez + \Lambda \sqrt{z^2 - 4ra_1(s)} + g(z; s) \right) dz. \quad (52)$$

The linear term in (52) will give no contribution by Cauchy theorem. Now we need to estimate term depending on g . Since g is analytic in a neighborhood of $\gamma(s)$, we may have the bound

$$\frac{1}{2\pi t_0(s)} \left| \oint_{\gamma(s)} g(z; s) dz \right| \leq \frac{\max_{z \in \gamma(s)} |g(z; s)|}{2\pi t_0(s)} \oint_{\gamma(s)} dz. \quad (53)$$

Since arc length $\oint_{\gamma(s)} dz$ is finite, $\max_{z \in \gamma(s)} |g(z; s)| = o(s)$, and by (16) and conditions (ii) of *Remark 3*.

$$t_0(s) = r^2 - |a_1(s)|^2 + o(s^2) = s + o(s),$$

we have that the r.h.s. of (53) is $o(s)/t_0(s) = o(1)$. Hence, $\Lambda/t_0(s) = 1/2r^2 + \mathcal{O}(s)$. Also, for $z \in \gamma(s)$,

$$\frac{\sqrt{4ra_1(s) - z^2}}{ra_1(s)} = \frac{\sqrt{4r^2 - z^2}}{r^2} + \mathcal{O}(s).$$

Combining the estimates we have Therefore,

$$I(s) = \frac{-1}{4\pi} \oint_{\gamma(s)} f(z) \frac{\sqrt{4r^2 - z^2}}{r^2} dz + o(1) \quad (54)$$

By using Proposition (2), $\gamma(s)$ may be deformed until it coincides with the branch cut of the square root. Because the branch the integrand along the around the cut will not be the same. In the first term running from $2r$ to $-2r$, the square root becomes $\sqrt{4r^2 - x^2}$, while in the second running from $-2r$ to $2r$ the square root becomes $-\sqrt{4r^2 - x^2}$. This leads to

$$I(s) = \frac{1}{2\pi r^2} \int_{-2r}^{2r} f(x) \sqrt{4r^2 - x^2} dx + o(1)$$

concluding the proof. \square

6.1 Proof of Theorem 1:

We have constructed a one parameter family of curves $\gamma(s)$ parametrized by $h(w; s)$ with $|w| = 1$. We have shown, in Proposition 1, that under the hypothesis (23) the family $\gamma(s)$ is composed of simple closed analytic polynomial curves of degree n . Moreover, $h(w; s)$ acts as the Riemann map from the exterior of the unity circle onto the exterior of $\gamma(s)$ for $s \in (0, 1]$. By Proposition 3, the problem of determining the exterior moments t_j out of simple closed analytic polynomial curves has a unique solution also when t_2 tends to $1/2$.

The balayage techniques enables us to sweep all the eigenvalues to the boundary $\gamma(s)$ of the support $D(s)$ and analyze the deformation only focusing on $\gamma(s)$. In Theorem 4 we deduce an

explicit equation relating the balayage measure with the potential V for $f \in \mathcal{H}$. It turns out that the balayage measure is proportional to the Schwarz function. Our focus changes to the behavior of the Schwarz functions for polynomial curves.

In Proposition 4 we show that the Schwarz function of every simple closed polynomial curve must have a branch cut which lies in the interior of the curve and the branch points never touch the curve itself. The limit $s \rightarrow 0$, the curve γ is smashed into the line, see Proposition 1. The same limit is analyzed in Lemma 4 where we show that the Schwarz function converges to a function of the type $\sqrt{r^2 - x^2}$.

This implies that as $s \rightarrow 0$ the normal ensemble converges to a Hermitian one (all the eigenvalues are real). Using these results and comparing the left hand side of (50) with the semicircle law (Eq. 2), we conclude that conformal deformations of Elbau-Felder ensembles to the real line yields a Wigner ensemble, as claimed. \square

7 Examples

7.1 Potential of degree 2

Taking $n = 1$ in (6), the potential V reads

$$V(z) = \frac{z\bar{z} - t_2 z^2 - \bar{t}_2 \bar{z}^2}{t_0}.$$

Consider the one parameter family for Riemann maps $h(w; s) = rw + a_1(s)w^{-1}$ in which, without loss of generality, r and $a_1(s) = r(1 - s)$ for $s \in (0, 1]$. Note that this parametrization fulfills the conditions (i - iii) by taking $s \mapsto s/2$. This yields $h(w; s) = r(w + (1 - s)w^{-1})$ $t_0 = r^2(2s - s^2)$ and $2t_2 = 1 - s$, by relations (16) and (17). The support of the eigenvalues is then given by

$$D(s) = \left\{ x + iy \in \mathbb{C} : \frac{x^2}{r^2(2 - s)^2} + \frac{y^2}{r^2 s^2} \leq 1 \right\}.$$

with major and minor semi-axis given, respectively, by $(2 - s)$ and s . For every $s \in (0, 1]$ the Elbau-Felder conditions is satisfied, namely, $t_1(s) = 0$ and $|t_2(s)| < 1/2$. Condition (23): $\xi(s) = r - a_2(s) = s > 0$ for $s \in (0, 1]$, holds and conditions (i - iii) are trivially satisfied.

The equilibrium measure associated to this potential is uniform within the support $D(s)$ and may be continuous deformed up to the limit $\lim_{s \rightarrow 0} h(w; s) = 2\Re(w)$ for $|w| = 1$, in which its boundary becomes the line segment $[-2r, 2r]$.

To show that the balayage measure converges to the Wigner measure, we consider the Schwarz function, which can be computed using the formulae in Ref. [18] or using $H(w; s)$ together with Proposition 5. By equation (46) and the Cauchy theorem, only the non-holomorphic part of the

Schwarz function gives a contribution to the integrals. For test functions $f \in \mathcal{H}(D(s))$ and the properties of the balayage measure (21), we have

$$\int_{D(s)} f(z) d\mu(z; s) = \frac{1}{2\pi r^2 (1-s)} \int_{-2r\sqrt{1-s}}^{2r\sqrt{1-s}} f(x) \sqrt{4r^2(1-s) - x^2} dx$$

It is also important to mention that during the whole process of conformal deformation of the normal ensemble to the Hermitian, the logarithm potential along the curve is kept unchanged, being an invariant in the process of deformation. One can see this by computing the integral explicitly.

7.2 Potentials of Degree 3: Breaking Down the Hypotheses

If we break the conditions (i – iii) of *Remark 3* ($\Delta_j > 1$), then it is impossible to deform the normal ensemble into a Hermitian one keeping, at same time, the regularity of the curve $\gamma(s)$ and a non trivial support. We shall illustrate this scenario for $n = 2$ taking $t_1 = t_2 = 0$. The potential then reads

$$V(z) = \frac{z\bar{z} - t_3 z^3 - \bar{t}_3 \bar{z}^3}{t_0}.$$

By (16) we have $t_0(s) = r^2 - 2|a_2(s)|^2$ and $3t_3(s) = a_2(s)/r^2$. The Riemann map is written as $h(w; s) = rw + a_2(s)w^{-2}$. Note that $a_1 = 0$ violating condition (ii). Therefore, the deformation of $\gamma(s)$ to a segment of the real line is impossible. The regularity of $\gamma(s)$, which is guaranteed by condition (23), requires $|a_2| < r/2$, while the positive area condition on t_0 requires $|a_2| < r/\sqrt{2}$.

There is no parametrization that keeps the regularity of γ . When the area πt_0 of $D(s)$ converges to zero ($a_2 \rightarrow r/2$), $\gamma(s)$ develops cusps and is no longer regular. Keeping the regularity of the curve, the limit $t_0 \rightarrow 0$ is possible only taking $r \rightarrow 0$. Therefore, as the area converges to zero γ collapses to a point.

8 Conclusion

We study the conformal deformations of Elbau-Felder ensembles to Hermitian ensembles. Special attention is paid to Wigner ensembles, for which the density of eigenvalues follows the semicircular law. The result presented in Theorem 1 is in a way universal, in the sense that it does not depend on the initial ensemble one starts with. Two ingredients are used to prove the main result. We use bifurcation theory from a single eigenvalue for extending the moment problem to near slit domains and the concept of balayage measure and Schwarz function for assisting the convergence of normal ensembles to Wigner ensembles.

It would be interesting to analyze the deformation of the Elbau-Felder ensembles to the real line without the assumption on the convergence rate. In this case the resulting measure, if well defined,

might deviate from the semicircular law, and it might also depend on the initial ensemble under consideration.

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A Solution of (31)

Lemma 2 *The linear system of equations (31) has a unique solution $\varphi = T\mathbf{v} + B\bar{\mathbf{v}}$ where $B = (1 - |k|^2)^{-1}JK$ with $k = K_{11}$ and $T = J + \bar{k}B$, provided $|k| \neq 1$.*

Proof. Equation (31) is solvable if and only if

$$\begin{pmatrix} B & T \\ \bar{T} & \bar{B} \end{pmatrix} \begin{pmatrix} -\bar{K} & J^{-1} \\ J^{-1} & -K \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

holds for some $(n+1) \times (n+1)$ complex matrices B and T and this is equivalent to

$$\begin{aligned} -B\bar{K} + TJ^{-1} &= I \\ BJ^{-1} - TK &= 0. \end{aligned} \tag{55}$$

Let us assume that B has its first column given by \mathbf{b} and 0 everywhere else (so B has the same form of K) and let $T = J + \bar{k}B$. Note that $KA = a_{11}K$ ($BA = a_{11}B$) holds for any matrix $A = [a_{ij}]$. Substituting T in (55), we have

$$\begin{aligned} BJ^{-1} - TK &= BJ^{-1} - (J + \bar{k}B)K \\ &= B - JK - |k|^2 B = 0 \end{aligned}$$

which implies

$$\begin{aligned} B &= \frac{1}{1 - |k|^2} JK \\ T &= J + \frac{\bar{k}}{1 - |k|^2} JK. \end{aligned}$$

and concludes the proof of the lemma. The uniqueness follows by linearity. \square

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